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LAMT 2026, May 17, 2026 - Guts Round Set 1

Team Name: ________________________________ Team ID: ____________

1. **[9]** Find the number of zeroes in the decimal representation of

$$\frac{1111111111111111}{10001 \cdot 101 \cdot 11},$$

(where there are 16 ones in the numerator).

Solution: $\boxed{7}$

Note that there are 16 ones. First divide by 11 to get

$$\frac{1111111111111111}{11} = 101010101010101$$

(which has 15 digits). Then divide by 101 to get

$$\frac{101010101010101}{101} = 1000100010001.$$

Finally divide by 10001 to get

$$\frac{1000100010001}{10001} = 100000001,$$

which has $\boxed{7}$ zeroes in it.

2. **[9]** Two real numbers b and c are chosen independently and uniformly at random from the interval $[0, 2]$. Find the probability that the quadratic $x^2 + bx - c^2 + 1 = 0$ has at least one real solution for x.

Solution: $\boxed{1 - \frac{\pi}{8}}$

We note that the quadratic $x^2 + bx - c^2 + 1$ has real zeroes if and only if the discriminant is nonnegative, or equivalently, $b^2 + 4c^2 \geq 4$. By inspection, it can be seen that the graph of this inequality describes the complement of a filled quarter-ellipse centered at the origin with a semi-major axis parallel to the x axis and of length 2, and a semi-minor axis parallel to the y axis and of length 1. Thus, the desired probability p is given by

$$p = \frac{4 - \pi/2}{4} = \boxed{1 - \frac{\pi}{8}}.$$

3. **[9]** A right triangle has integer leg lengths, and a hypotenuse of length $2\sqrt{2026}$. Find the sum of the lengths of the legs.

Solution: $\boxed{92}$

Recall that $2025 = 45^2$ is a perfect square, meaning $45^2 + 1^2 = 2026$. Using this, we find $8104 = 90^2 + 4^2$, so the sum of the legs is $90 + 2 = \boxed{92}$. There are no other positive integer solutions.

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LAMT 2026, May 17, 2026 - Guts Round Set 2

Team Name: ________________________________ Team ID: ____________

4. **[10]** Let $f(n)$ denote the number of letters in the standard American spelling of n. For example $f(3) = 5$ since "three" has 5 letters in it. Find

$$f(f(f(f(1)))) + f(f(f(f(2)))) + \cdots + f(f(f(f(20)))).$$

Solution: $\boxed{80}$

All the numbers below 10 map to 3, 4 or 5. 3 maps to 5, which maps to 4, which is a fixed point. So after 3 applications of f, these will all be 4. For numbers above 10, they will all map to something below 10 with one application of f, so after 3 more they will all map to 4. Thus the answer is $4 \cdot 20 = \boxed{80}$.

5. **[10]** There exists a unique ordered pair (a, b) of positive integers satisfying the following: $a < b$, $a + b = 801$, and the second largest divisor of a plus the second largest divisor of b is 337. Find the ordered pair (a, b).

Solution: $\boxed{(381, 420)}$

First since a and b sum to an odd integer, one of them must be odd and the other must be even. Without loss of generality a is even (ignore $a < b$ for now). Then the second largest divisor of b is $\frac{b}{p}$ where $p > 2$ is the smallest prime factor of b. Thus we have

$$a + b = 801 \text{ and } \frac{a}{2} + \frac{b}{p} = 337.$$

Then substituting $a = 801 - b$, we get

$$\frac{801 - b}{2} + \frac{b}{p} = 337 \implies b(p - 2) = (801 - 2 \cdot 337)p = 127p.$$

Then clearly 127 must divide b, and $\frac{b}{127} = \frac{p}{p-2}$. The left hand side is an integer, so $\frac{p}{p-2}$ is an integer, or $p = 3$. Then we can easily solve to get $(a, b) = \boxed{(381, 420)}$.

6. **[10]** Let $P(x)$ be a polynomial with nonnegative integer coefficients for which $P(1) = 10$ and $P(10) = 2026$. Find $P(2)$.

Solution: $\boxed{26}$

Note that since $P(1) = 10$, 10 must be greater than or equal to all of the coefficients of $P(x)$. If one of the coefficients is 10, then the polynomial is of the form $10x^n$, which doesn't work as $P(10) = 2026$ is not divisible by 10. Thus all coefficients of P are strictly less than 10.

So when we plug 10 into $P(x)$, the digits of $P(10)$ must be the coefficients of $P(x)$. Thus $P(x) = 2x^3 + 0x^2 + 2x + 6$, so $P(2) = \boxed{26}$.

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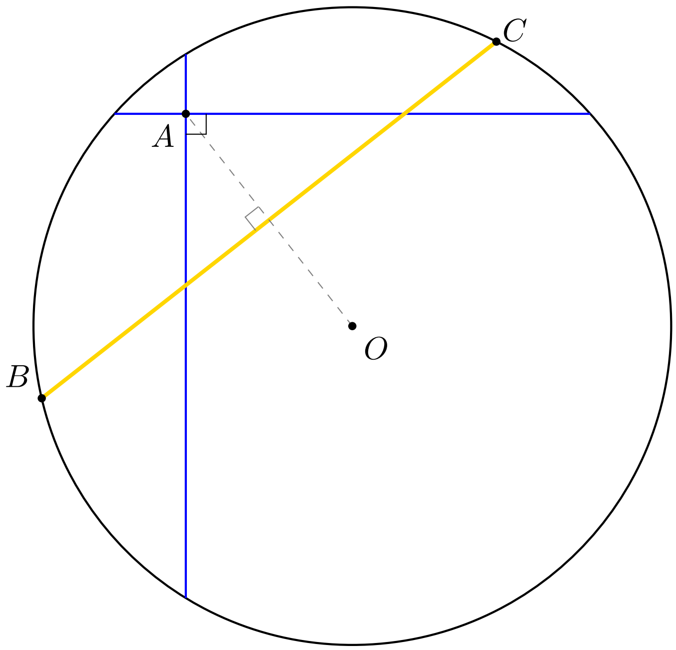
LAMT 2026, May 17, 2026 - Guts Round Set 3

Team Name: ______________________________ Team ID: ___________

7. **[12]** Two chords of a circle with center O have lengths 14 and 16, are perpendicular, and intersect at A. The perpendicular bisector of OA meets the circle at two points B and C such that $BC = 17$. Find the radius of the circle.

Solution: $\boxed{2\sqrt{22}}$



Let the radius of the circle be r. Then the distances from O to the chords of length 14 and 16 are $\sqrt{r^2-49}$ and $\sqrt{r^2-64}$, respectively. Then by the Pythagorean Theorem,

$$OA^2 = r^2 - 49 + r^2 - 64 = 2r^2 - 113.$$

Additionally the distance from O to the chord of length 18 is half of OA, so

$$OA^2 = 4\left(r^2 - \frac{289}{4}\right).$$

Then we can solve for r:

$$2r^2 - 113 = 4r^2 - 289 \implies r^2 = \boxed{2\sqrt{22}}.$$

8. **[12]** There is a unique polynomial $P(x)$ such that for every positive integer i,

$$P(i) = \sum_{j=1}^{i} \sum_{k=1}^{j} ijk.$$

Find the value of $P(-3)$.

Solution: $\boxed{-6}$

First of all, let $Q(x)$ be the unique polynomial such that $Q(i) = \sum_{j=1}^{i} \sum_{k=1}^{j} jk$ for all $i \in \mathbb{Z}^+$. Then $P(i) = iQ(i)$ for all $i \in \mathbb{Z}^+$, so $P(x) = xQ(x)$ for all $x \in \mathbb{R}$ because both sides of the equation are polynomials that agree at infinitely many points.

Now let $R(x)$ be the unique polynomial such that $R(j) = \sum_{k=1}^{j} jk$ for all $j \in \mathbb{Z}^+$. This defining property means that $Q(i) = \sum_{j=1}^{i} R(j)$ for all $i \in \mathbb{Z}^+$. Also, for every $j \in \mathbb{Z}^+$,

$$\begin{aligned} R(j) &= \sum_{k=1}^{j} jk \\ &= j \sum_{k=1}^{j} k \\ &= j \frac{(j+1)j}{2} \\ &= \frac{(j+1)j^2}{2}. \end{aligned}$$

Thus, $R(x) = (x+1)x^2/2$ for all $x \in \mathbb{R}$, because both sides of the equation are polynomials that agree at infinitely many points.

Now, for all $i \in \mathbb{Z}^+$, we can use $Q(i) = \sum_{j=1}^{i} R(j)$ to find that $Q(i+1) = Q(i) + R(i+1)$. Then $Q(x+1) = Q(x) + R(x+1)$ for all $x \in \mathbb{R}$, as both sides of this equation are polynomials that agree at infinitely many points. Rearranging, $Q(x) = Q(x+1) - R(x+1)$.

Now, we can start with $Q(1) = \sum_{j=1}^{1} \sum_{k=1}^{j} jk = 1$ and evaluate $Q(x)$ at incrementally lower values of x:

$$\begin{aligned} Q(0) &= Q(1) - R(1) \\ &= 1 - 1 = 0 \\ Q(-1) &= Q(0) - R(0) \\ &= 0 - 0 = 0 \\ Q(-2) &= Q(-1) - R(-1) \\ &= 0 - 0 = 0 \\ Q(-3) &= Q(-2) - R(-2) \\ &= 0 - \frac{(-1) \cdot (-2)^2}{2} \\ &= 2. \end{aligned}$$

Hence, $P(-3) = (-3)Q(-3) = \boxed{-6}$.

9. **[12]** Let $x^3 - 9x^2 + 5x - 1$ have roots a, b, and c. Find

$$\left(\frac{1}{a} + \frac{1}{b} + 1 + c\right)\left(\frac{1}{b} + \frac{1}{c} + 1 + a\right)\left(\frac{1}{c} + \frac{1}{a} + 1 + b\right).$$

Solution: $\boxed{256}$

Note that $abc = 1$, so we can rewrite

$$\begin{aligned} \left(\frac{1}{a} + \frac{1}{b} + 1 + c\right) &= \left(\frac{1}{a} + \frac{1}{b} + 1 + \frac{1}{ab}\right) \\ &= \frac{b + a + ab + 1}{ab} \\ &= \frac{(a+1)(b+1)}{ab}. \end{aligned}$$

This our product is

$$\frac{((a+1)(b+1)(c+1))^2}{(abc)^2} = ((a+1)(b+1)(c+1))^2 = P(-1)^2 = \boxed{256}.$$

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LAMT 2026, May 17, 2026 - Guts Round Set 4

Team Name: ________________________________ Team ID: ____________

10. **[14]** Find the unique two digit positive integer x satisfying the following: x^2 is the four-digit number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$, and the two-digit numbers $\underline{a}\,\underline{b}$ and $\underline{c}\,\underline{d}$ sum to 121.

    Solution: $\boxed{88}$

    Let $m = \underline{a}\,\underline{b}$ and $n = \underline{c}\,\underline{d}$. Then $121 = m + n$ and $x^2 = 100m + n$. Then $x^2 - 121 = 99m$, so x is divisible by 11. Then let $x = 11k$, so $x^2 = 121k^2$. Taking modulo 9, we have $121k^2 \equiv \underline{a}\,\underline{b}\,\underline{c}\,\underline{d} \equiv \underline{a}\,\underline{b} + \underline{c}\,\underline{d} \equiv 121 \pmod 9$, so $k^2 \equiv 1 \pmod 9$. The only possible k is $k = 8$, so $x = \boxed{88}$.

11. **[14]** Andrew draws positive integers at most 100 uniformly at random (without replacement), stopping once he has drawn all the multiples of 6. Find the probability he draws all the multiples of 5.

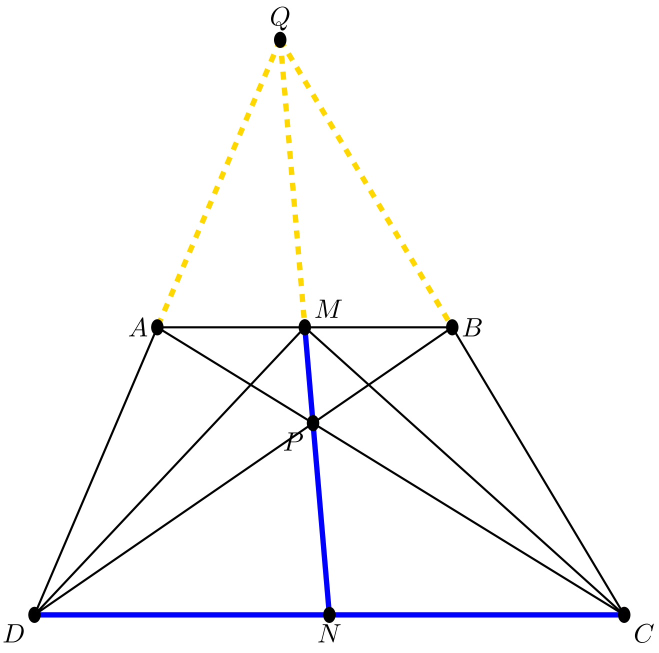
    Solution: $\boxed{\frac{16}{33}}$

    Discard numbers which are neither multiples of 6 nor multiples of 5. There are 3 numbers which are multiples of both 5 and 6 (denote these by A). There are 20 multiples of 5, 17 of which are only divisible by 5 (denote these by B). There are 16 multiples of 6, 13 of which are only divisible by 6 (denote these by C).

    Thus we can think of this process as arranging 3 As, 17 Bs, and 13 Cs. The requirement is that the last number is either an A or a C. This happens with probability $\frac{13+3}{13+3+17} = \boxed{\frac{16}{33}}$.

12. **[14]** Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 5$, $BC = 6$, and $CD = 10$. Let the midpoint of AB be M, and suppose $\angle CMD = 90^\circ$. Let AC and BD intersect at P. Find PM.

    Solution: $\boxed{\frac{5}{3}}$



Extend BC and DA to meet at Q, and let N be the midpoint of CD. Then note that P is the centroid of DQC, so Q, M, and P are collinear. Let N be the midpoint of CD. Additionally since $\angle DMC = 90^\circ$, $MN = 5$. Since M is the midpoint of NQ, $QN = 10$. Then $PN = \frac{10}{3}$, so the answer is $5 - \frac{10}{3} = \boxed{\frac{5}{3}}$.

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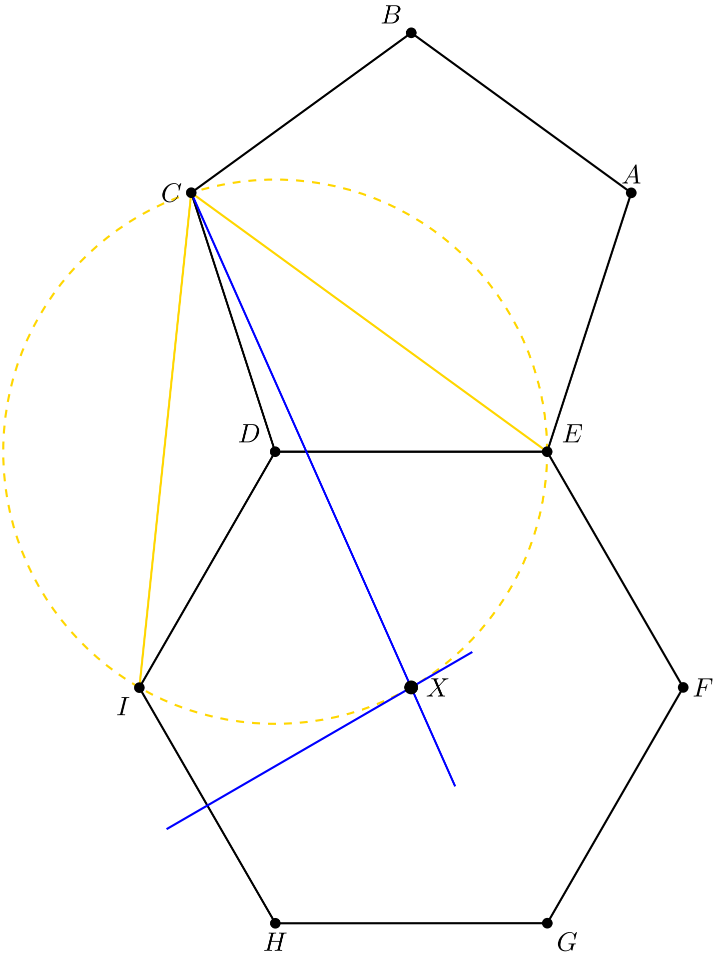
LAMT 2026, May 17, 2026 - Guts Round Set 5

Team Name: ________________________________ Team ID: ___________

13. **[17]** Let $ABCDE$ be a regular pentagon and $DEFGHI$ a regular hexagon which don't overlap. The angle bisector of $\angle ICE$ intersects the perpendicular bisector of IH at X. Find $\angle AXE$, in degrees.

    Solution: $\boxed{6}$



Let O be the center of $DEFGHI$. Note that D is the circumcenter of ICE as $DC = DE = DI$. Then the angle bisector of $\angle ICE$ intersects the midpoint of arc IE, which is O. But O also lies on the perpendicular bisector of IH, so $O = X$.

To finish note that $AE = EO$, and $\angle AEO = 108 + 60 = 168$. Then $\angle AXE = \frac{180-168}{2} = \boxed{6}$.

14. **[17]** Find the number of ordered triples (a, b, c) of integers in $\{0, 1, 2, \ldots, 25\}$ for which

$$20 \cdot 26 \leq a + 20b + 26c < 26^2.$$

Solution: $\boxed{4056}$

Let $k = a+20b+26c$. Then in fact, given any value of b in $\{0, 1, \ldots, 25\}$ and a value of k with $20\cdot 26 \leq k < 26^2$, we can retrieve a unique pair (a, c). This is because $a+26c$ is a calculation in base 26, since a and c are limited to the numbers $\{0, 1, \ldots, 25\}$. We see k will never be too large, since $a + 26c$ achieves any value up to $26^2 - 1$.

Additionally, $20b \leq 500 < 20 \cdot 26$, so k will never be too small. The answer is then $(26^2 - 20 \cdot 26) \cdot 26 = \boxed{4056}$.

15. **[17]** Let a_1, a_2, $\ldots$, a_{16} be positive integers less than or equal to $17 \cdot 19$. Suppose for each r in $\{1, 2 \ldots, 16\}$, there exist m and n in $\{1, 2, \ldots, 16\}$ for which 17 divides $a_m - r$ and 19 divides $a_n - r$. Find the maximum possible value of $a_1 + a_2 + \cdots + a_{16}$.

    Solution: $\boxed{4981}$

    Observe that the total sum of the numbers mod $17 \cdot 19$ is fixed, as it is fixed modulo each of 17 and 19. This sum is exactly

    $$1 + 2 + \cdots + 16 = 136.$$

    Additionally each of the 16 numbers is at most $17 \cdot 19$, so the answer is at most $17 \cdot 19 \cdot 16$. The largest number satisfying these properties is

    $$136 + 17 \cdot 19 \cdot 15 = \boxed{4981}.$$

    Thus it suffices to show this is achievable. This works by setting the residues equal to

    $$(14, 16), (13, 15), \ldots, (1, 3)$$

    with the last two being $(15, 2)$ and $(16, 1)$. Here the first coordinate represents value modulo 17, and the second modulo 19. This is natural to try, as the large residues near 323 are mainly of the form $(x, x + 2)$.

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LAMT 2026, May 17, 2026 - Guts Round Set 6

Team Name: ______________________________ Team ID: ____________

16. **[18]** Consider the monic polynomial $f(x)$ with minimum degree satisfying

    $$\frac{f(0)}{f(1)} = \frac{f(2)}{f(3)} = \frac{f(4)}{f(5)} = \frac{1}{2}.$$

    Find $f(6)$.

    Solution: $\boxed{141}$

    Consider $-2f(x) + f(x+1) = g(x)$, so $g(x)$ has roots $0, 2, 4$. Thus, $-2f(x) + f(x+1) = -x(x-2)(x-4) = -x^3 + 6x^2 - 8x$ Let $f(x) = x^3 + ax^2 + bx + c$, we then find $-2f(x) + f(x+1) = -x^3 + (3-a)x^2 + (3 + 2a - b)x + (1 + a + b - c), a = -3, b = 5, c = 3,\ f(6) = \boxed{141}$.

17. **[18]** Over all integers $k \geq 1$, find the number of tuples $(a_1, a_2, \ldots a_k, b_1, b_2, \ldots b_k)$ of positive integers satisfying $a_1 = 1$, $a_{i+1} = a_i(b_i + 1)$, and

    $$\sum_{i=1}^{k} a_i b_i = 383.$$

    Solution: $\boxed{576}$

    Rewriting everything in terms of b_i we get

    $$\begin{aligned} 383 = \quad & b_1 + (b_1 + 1)(b_2) \\ & + (b_1 + 1)(b_2 + 1)(b_3) \\ & \ldots \\ & + (b_1 + 1)(b_2 + 1) \ldots (b_{k-1} + 1)(b_k) \end{aligned}$$

Adding 1 to both sides the b_1 becomes $b_1 + 1$ and combines with $(b_1 + 1)(b_2)$ to become $(b_1 + 1)(b_2 + 1)$, which then combines with the next term $(b_1 + 1)(b_2 + 1)(b_3)$ to get $(b_1 + 1)(b_2 + 1)(b_3 + 1)$, and so on. Thus

$$384 = \prod_{i=1}^{k}(b_i + 1).$$

Then it suffices to find the number of ways to multiply k positive integers ≥ 1 to get $384 = 2^7 \cdot 3$.

To do this imagine lining up seven 2s and then splitting them up into k pieces. There are $\binom{6}{k-1}$ ways to do this. Then we can put the 3 with any of these powers of 2, or in any of the $k + 1$ gaps. There are $2k + 1$ options, so the answer is

$$\sum_{k=1}^{7} \binom{6}{k-1}(2k + 1).$$

By pairing k and $6 - k$ this can be easily computed to be

$$\frac{1}{2}\sum_{k=0}^{6}\binom{6}{k}(2k + 3 + 2(6 - k) + 3) = \frac{1}{2}\sum_{k=0}^{6} 18\binom{6}{k} = \frac{1}{2} \cdot 18 \cdot 2^6 = \boxed{576}.$$

18. **[18]** Call an n digit positive integer $\underline{a_1a_2a_3 \ldots a_n}$ *six-seven divisible* if there exists a positive integer $k < n$ such that 6 divides $\underline{a_1a_2a_3 \ldots a_k}$ and 7 divides $\underline{a_{k+1}a_{k+2} \ldots a_n}$. For example, 628 and 720 are six-seven divisible (6 divides both 6 and 72, and 7 divides both 28 and 0), while 362 and 90 are not. Find the number of four-digit six-seven divisible positive integers with a nonzero leading digit.

Solution: $\boxed{619}$

Let S_k be the set of six-seven divisible numbers with $\underline{a_1a_2 \ldots a_k}$ divisible by 6. By the Principle of Inclusion-Exclusion, our answer is equal to:

$$|S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_3 \cap S_1| + |S_1 \cap S_2 \cap S_3|$$

For $k = 1$, $a_1 = 6$, leaving $\underline{a_2a_3a_4}$ needing to just be a multiple of 7 less than 1000. There are 143 possible values here. For $k = 2$, $\underline{a_1a_2}$ could equal 12, 18, 24, ..., 96, and $\underline{a_3a_4}$ could equal any multiple of 7 less than 100. There are $15 \cdot 15 = 225$ possible values here. For $k = 3$, $\underline{a_1a_2a_3}$ could equal 102, 108, 114, ..., 996, and a_4 equals 0 or 7 There are $150 \cdot 2 = 300$ possible values here. Numbers in $S_1 \cap S_2$ require $\underline{a_1a_2} = 60$, and $\underline{a_3a_4}$ could equal any multiple of 7 less than 100, so there are 15 cases here. Numbers in $S_2 \cap S_3$ require $a_3 = 0$, $\underline{a_1a_2}$ to be divisible by 6, and a_4 to be either 0 or 7. There are $15 \cdot 2 = 30$ cases here. Numbers in $S_3 \cap S_1$ require $a_1 = 6$, $\underline{a_2a_3}$ to be divisible be divisible by 42, and a_4 to be either 0 or 7. There are $3 \cdot 2 = 6$ cases here. Numbers in $S_1 \cap S_2 \cap S_3$ require $\underline{a_1a_2a_3} = 600$, and a_4 to be either 0 or 7. There are 2 cases here. Our answer is thus:

$$143 + 225 + 300 - 15 - 30 - 6 + 2 = \boxed{619}.$$

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LAMT 2026, May 17, 2026 - Guts Round Set 7

Team Name: ______________________________ Team ID: ____________

19. **[21]** The numbers 1, 2, 3, and 4 are placed in a 4×4 grid such that each cell has exactly one number, and each number appears exactly 4 times. Two cells are called *friends* if they have the same entry and are in the same row or column. Find the number of arrangements for which each cell has exactly two friends.

Solution: $\boxed{1080}$

The condition of each cell having two friends is equivalent to each number forming an axis aligned rectangle. First place the rectangle for the 1s, which has $\binom{4}{2}^2 = 36$ options. For the purpose of easy visualization, without loss of generality this is the 2×2 grid in the top left. We have two cases:

Case 1: The other numbers are in the top right 2×2, bottom right 2×2, and bottom left 2×2 grid. There are 6 ways to assign this.

Case 2: The other numbers form some non-contiguous rectangles. By inspection, we see that two of the numbers are either vertically or horizontally staggered. There are two ways to choose the side, and then two ways to stagger them. Then there are 6 ways to assign the numbers. This gives 24 cases.

In total the answer is $36 \cdot (6 + 24) = \boxed{1080}$.

20. **[21]** Find the smallest positive integer n for which $n \equiv 100 \pmod{101^2}$ and $n \equiv 101 \pmod{100^2}$.

Solution: $\boxed{9999^2 + 100}$

Replace 100 by k. Then in general the answer is $k + (k^2 - 1)^2$.

The first condition allows to set $n = k + m(k+1)^2$ for some positive integer m. Now taking modulo k^2 we get

$$k + 1 \equiv k + m(k+1)^2 \pmod{k^2} \implies 1 \equiv m(k+1)^2 \pmod{k^2}.$$

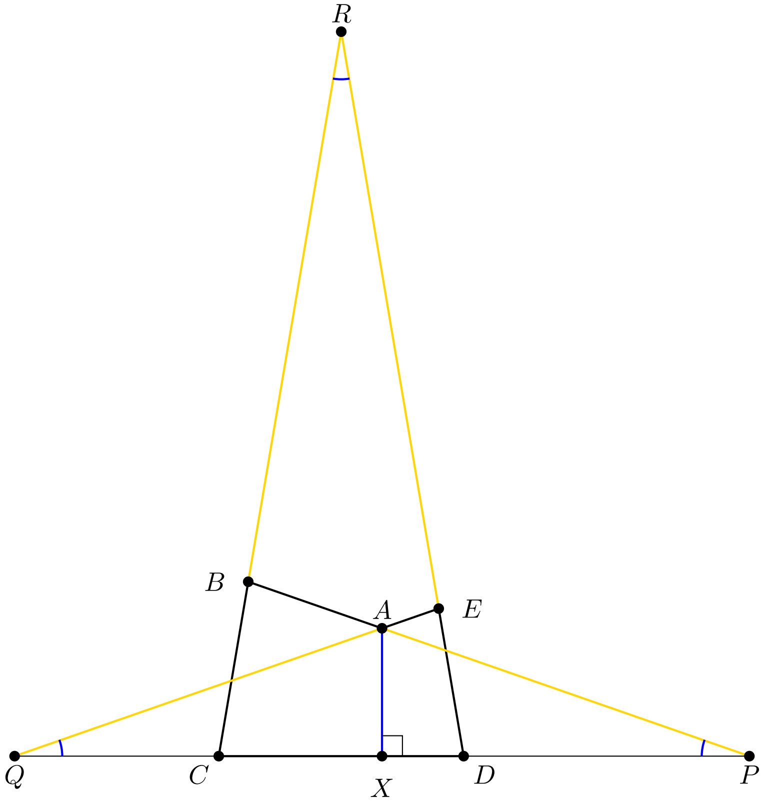
Now motivated by the fact that $(k+1)(k-1) = k^2 - 1$, we note that

$$((k+1)(k-1))^2 \equiv (-1)^2 \equiv 1 \pmod{k^2}.$$

Thus m is actually $(k-1)^2$. Thus $n = k + (k+1)^2(k-1)^2 = k + (k^2-1)^2$ is a solution. It is the smallest solution as there is a unique solution modulo $k^2(k+1)^2$ which is larger than our value of n. Plugging in $k = 100$ we get $9999^2 + 100$.

21. **[21]** Let $ABCDE$ be a pentagon with $\angle A > 180°$ and $\angle B = \angle C = \angle D = \angle E$. The angle bisector of $\angle A$ meets CD at X. Given $BC = 13$, $DE = 11$, $CX = 12$, and $XD = 6$, find $\frac{AB}{AE}$.

Solution: $\boxed{\frac{59}{25}}$



Extend AB and AE past A to meet CD at P and Q, respectively. Also extend CB and ED to meet at R. Then note that BPC, EQD, and CRD are isosceles and are all similar.

Then let $PC = 13x$, so $QD = 11x$. Note that $\angle APQ = \angle AQP$, so AQP is isosceles. We see that AX must be perpendicular to CD, so $QX = XP$. Then

$$11x - 6 = QX = XP = 13x - 12 \implies x = 3.$$

Since $x = 3$ and $CR = (12 + 6)x$, we see $CR = DR = 54$. Then $BR = 41$ and $ER = 43$. Note that AB and AE form the same angle with CD, so we just consider the displacement of these along the path from C to R to D.

B has a displacement of 13, X has a displacement of $54 \cdot 2 \cdot \frac{12}{12+6} = 72$. E has a displacement of $13+41+43 = 97$. The answer is then $\frac{72-13}{97-72} = \boxed{\frac{59}{25}}$.

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LAMT 2026, May 17, 2026 - Guts Round Set 8

Team Name: ______________________________ Team ID: ____________

22. **[24]** Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the number of positive integers which can be expressed in the form

$$\lfloor \alpha \rfloor + \lfloor \alpha^2 \rfloor + \cdots + \lfloor \alpha^{10} \rfloor$$

for some real number $1 < \alpha \leq 2$.

Solution: $\boxed{1966}$

Whenever $\lfloor \alpha^n \rfloor$ increases by 1, so does $\lfloor \alpha^k \rfloor$ for all $k \mid n$. Additionally $\lfloor \alpha^n \rfloor$ increases by 1 exactly 2^n times. If we add

$$2^{10} + 2^9 + 2^8 + 2^7 + 2^6,$$

then this counts the increases for $\lfloor \alpha^{10} \rfloor$, $\lfloor \alpha^9 \rfloor$, $\lfloor \alpha^8 \rfloor$, $\lfloor \alpha^7 \rfloor$, $\lfloor \alpha^6 \rfloor$, $\lfloor \alpha^5 \rfloor$, and $\lfloor \alpha^4 \rfloor$ exactly once. But the increase for $\lfloor \alpha^3 \rfloor$ is counted 3 times (two for 6 and 9), $\lfloor \alpha^2 \rfloor$ is counted 3 times (for 10, 8, and 6) and $\lfloor \alpha^1 \rfloor$ is counted 5 times (for 10, 9, 8, 7, 6). If we subtract 2^3, then increases for $\lfloor \alpha^3 \rfloor$ are counted once, but increases for $\lfloor \alpha^1 \rfloor$ are now counted 4 times. Then we subtract $2^2 \cdot 2$ to count $\lfloor \alpha^2 \rfloor$ one time. Finally we subtract 2^1 to account for $\lfloor \alpha^1 \rfloor$. The answer is

$$2^{10} + 2^9 + 2^8 + 2^7 + 2^6 - 2^3 - 2^2 \cdot 2 - 2^1 = \boxed{1966}.$$

23. **[24]** There are 12 cards, each with a number from 1 to 6 so that each number appears twice. Yibo draws 6 of these cards uniformly at random, and gives the other 6 to Gautham. Starting with Yibo, each player plays the card in their hand with the lowest value that is at least as large as all previously played cards. The game ends once a player cannot play a card. Find the probability that exactly 6 cards are played.

Solution: $\boxed{\frac{11}{84}}$

First observe that if the two cards labeled with k are split across the two players, both of them must be placed. Furthermore, observe that the number of such k must be even. We then see that either 0 or 2 values of k have this property.

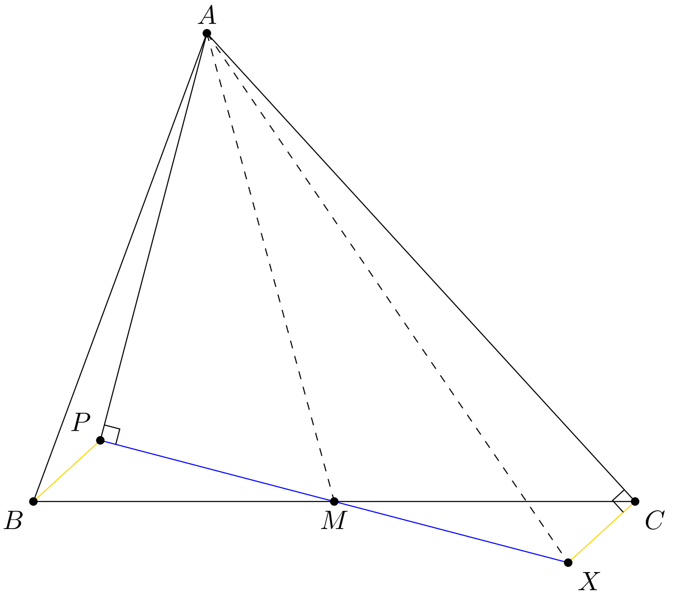
Case 1: None of the cards are split across the two players. This forces the first player to have both 1s, the second to have both 2s, and so on. This gives only 1 ways.

Case 2: Exactly two labels are split across the two players. There are $\binom{6}{2}$ ways to select the labels that are split, then 2^2 ways to choose who gets which card. These 4 cards will always be placed, and don't interact with the other cards. Order the remaining 4 labels $A < B < C < D$. We need to count the number of cases in which exactly two of these are drawn. Checking all $\binom{4}{2}$ cases, it is easy to see that the first player must have A and B, or they have B and C. Thus this case contributes $\binom{6}{2} \cdot 2^2 \cdot 2 = 120$ cases.

In total there are $\binom{12}{6}$ ways to assign the cards, giving us a probability of $\frac{1+120}{\binom{12}{6}} = \boxed{\frac{11}{84}}$.

24. **[24]** Let ABC be a triangle with M the midpoint of BC. A point P lies inside ABC with $\angle APM = 90°$ and $BP \perp AC$. Suppose $BP = 3$, $PM = 8$, and $AC = 21$. Find AM.

Solution: $\boxed{\sqrt{258}}$



Reflect P over M to a point X. Then note that $BPCX$ is a parallelogram, so $AC \perp BP \parallel XC$, meaning $\angle XCA = 90°$. Additionally $XC = 3$, $PX = 2 \cdot 8 = 16$, and $\angle APX = 90°$. Thus by the Pythagorean Theorem

$$AP^2 + PX^2 = AC^2 + XC^2 \implies AP^2 = 21^2 + 3^2 - (2 \cdot 8)^2 = 194.$$

Then by the Pythagorean Theorem on APM, we get $AM^2 = 194 + 8^2 = \boxed{\sqrt{258}}$.

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LAMT 2026, May 17, 2026 - Guts Round Set 9

Team Name: ________________________________ Team ID: ____________

25. **[25]** Let N be the number of 20-digit strings (each digit from 0 to 9) such that no two adjacent digits differ by more than 1, and the first digit is 5. Estimate $\log_{10}(N)$.

Submit a number E. If the true answer is A, you will receive $\max(0, \lfloor 26 - 500|E - A| \rfloor)$ points.

Solution: $\boxed{8.93543617}$

Given a valid string of length n, there are most likely 3 options for a digit to append to it, while maintaining a valid string. The exception is when the last digit is a 0 or 9. Using this, the answer is at most 3^{19}, and we can approximate $\log_{10}(3^{19}) = 19 \log_{10}(3) \approx 9$. The answer should be a little bit less than this.

26. **[25]** Positive reals $p_1,\ p_2,\ \ldots, p_{67}$ are randomly and independently generated such that, for each $n \in 1, 2, \ldots, 67$, the probability that $p_n = p$ for $p \in [1, \infty)$ is proportional to $\frac{7^7}{6!}(\ln(p))^6 p^{-8}$, and 0 for $p < 1$. Estimate the expected value of

$$\sum_{k=1}^{67} \log_{10}(p_k).$$

Submit a number E. If the true answer is A, you will receive $\min\left(25, \left\lfloor 25e^{-0.67|A-E|}\right\rfloor\right)$ points.

Solution: $\boxed{31.398}$

Note that

$$\sum_{k=1}^{67} \log(p_k) = \log\left(\prod_{k=1}^{67} p_k\right),$$

so we can find the expected value of $\prod_{k=1}^{67} p_k$.

Since each of the variables are independent, we can just find

$$\mathbb{E}[p] = \int_1^\infty p \cdot \frac{7^7}{6!}(\ln(p))^6 p^{-8}\, dp = \int_1^\infty \frac{7^7}{6!}(\ln(p))^6 p^{-7}\, dp,$$

and take that to the 67th power. Let $p = e^t$, then

$$\int_1^\infty \frac{7^7}{6!}(\ln(p))^6 p^{-7}\, dp = \int_0^\infty \frac{7^7}{6!} t^6 e^{-6t}\, dt = \frac{7^7}{6!} \cdot \frac{6!}{6^7} = \left(\frac{7}{6}\right)^7,$$

where the integral follows from using the Laplace Transform of t^6.

Thus,

$$\mathbb{E}\left[\prod_{k=1}^{67} p_k\right] = \left(\left(\frac{7}{6}\right)^7\right)^{67} = \left(\frac{7}{6}\right)^{469},$$

so our answer is $469\log(\frac{7}{6}) \approx \boxed{31.398}$

27. **[25]** Let $\tau(n)$ denote the number of positive divisors of n. Estimate the value of

$$\sum_{n=1}^{10^6} \frac{\tau(n)}{n}.$$

Submit a number E. If the true answer is A, you will receive $\max(0, \lfloor 26 - 4|E - A|\rfloor)$ points.

Solution: $\boxed{111.862126}$

This can be approximated quite accurately using Abel Summation Formula:

$$\sum_{n \le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\, dt$$

where

$$A(x) = \sum_{n \le x} a_n,$$

and we set $a_n = \tau(n)$. We can also set $f(t) = \frac{1}{t}$, which gives $f'(t) = -\frac{1}{t^2}$. For our choice of A, it is well known that

$$A(x) = x\log x + (2\gamma - 1)x + O(\sqrt{x})$$

We then compute

$$A(x)f(x) = \frac{x\log x + (2\gamma - 1)x + O(\sqrt{x})}{x} = \log x + (2\gamma - 1) + O(x^{-1/2})$$

The integral term becomes $\int_1^x \frac{A(t)}{t^2}\,dt$. Substituting the expansion for $A(t)$ yields:

$$\begin{aligned}\int_1^x \frac{t\log t + (2\gamma-1)t + O(t^{1/2})}{t^2}\,dt &= \int_1^x \frac{\log t}{t}\,dt + \int_1^x \frac{2\gamma-1}{t}\,dt + \int_1^x O(t^{-3/2})\,dt \\ &= \frac{1}{2}\log^2 x + (2\gamma-1)\log x + C' + O(x^{-1/2})\end{aligned}$$

where C' is a constant arising from the lower limit of integration and the convergent error term. Adding the two parts we get an estimate of

$$\begin{aligned}\sum_{n\le x}\frac{\tau(n)}{n} &= (\log x + 2\gamma - 1) + \left(\frac{1}{2}\log^2 x + (2\gamma-1)\log x\right) + C' \\ &\approx \frac{1}{2}\log^2 x + 2\gamma\log x + C + O(x^{-1/2})\end{aligned}$$

From here we approximate $\log 10 \approx 2.3$, so $\log 10^6 \approx 14$. Then an estimate of $\frac{1}{2}\cdot 14^2 + 14 = 112$ is quite close to the correct answer, and enough for full points.

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